Combinations of Random Variables

Previously, you have learnt how to solve problems involving random variables and normally distributions. In this chapter, you will learn how to linearly combine random variables which are normally distributed.

Combination of Random Variables

Previously, we have learnt that:

When combining two random variables, X and Y:

$$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$
$$\mathrm{E}(X-Y) = \mathrm{E}(X) - \mathrm{E}(Y)$$
$$\mathrm{E}(aX+bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$$
$$\mathrm{E}(aX-bY) = a\mathrm{E}(X) - b\mathrm{E}(Y)$$

For two random independent variables, X and Y:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$
$$\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$
$$\mathrm{Var}(aX+bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$
$$\mathrm{Var}(aX-bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

Note: the outcomes of independent variables are not affected by each other

Combination of Random Variables with Normal Distributions

You can also combine one or more random variables linearly with normal distribution. The combined variable will also have a normal distribution. The mean and standard deviation can be found using the following:

For the random independent variables $X \sim \mathrm{N}(\mu_1, \sigma_1{}^2)$ and $Y \sim \mathrm{N}(\mu_2, \sigma_2{}^2)$:

$$aX + bY \sim \mathrm{N}(a\mu_1 + b\mu_2, a^2\sigma_1{}^2 + b^2\sigma_2{}^2)$$
$$aX - bY \sim \mathrm{N}(a\mu_1 - b\mu_2, a^2\sigma_1{}^2 + b^2\sigma_2{}^2)$$

Example 1: The two independent random variables, X and Y have the distributions $X \sim \mathrm{N}(68, 5)$ and $Y \sim \mathrm{N}(82, 10)$. Given that the random variable $C = 2X + Y$.

i) Find the distribution of C.

Find $a\mu_1 + b\mu_2$.	$a\mu_1 + b\mu_2 = 2(68) + 82$ $= 218$
Find $a^2\sigma_1{}^2 + b^2\sigma_2{}^2$.	$a^2\sigma_1{}^2 + b^2\sigma_2{}^2 = 2^2(5)^2 + 10^2$ $= 200$
	The distribution of C is: $C \sim \mathrm{N}(218, 200)$

ii) Find $\mathrm{P}(2X + Y < 207)$

$\mathrm{P}(2X + Y < 207) = \mathrm{P}(C < 207)$.	$\mathrm{P}(C < 176) = \mathrm{P}(Z < \frac{207-218}{\sqrt{30}})$ $= \mathrm{P}(Z < -2.008)$ $= 1 - 0.9772$ $= 0.0228$

Example 2: The mean mass of apples is 125 grams and the standard deviation is 4 grams. The mean mass of oranges is 250 grams and the standard deviation is 5 grams. Each packet of mixed fruits contains 3 apples and 3 oranges. Given that the fruits were randomly selected and their masses are independent of each other, find the probability that a randomly selected packet weighs more than 1140 grams.

Assign variables to mass of apple, orange and packet of fruits.	X = mass of apple Y = mass of orange R = mass of packet of fruits $R = 3X + 3Y$
Write down the distribution of variables X and Y.	$X \sim \mathrm{N}(125, 4^2)$ $Y \sim \mathrm{N}(250, 5^2)$
Find the distribution of R.	$aX + bY \sim \mathrm{N}(a\mu_1 + b\mu_2, a^2\sigma_1{}^2 + b^2\sigma_2{}^2)$ $3X + 3Y \sim \mathrm{N}(3 \times 125 + 3 \times 250, 3^2(4^2) + 3^2(5^2))$ $3X + 3Y \sim \mathrm{N}(1125, 369)$ $R \sim \mathrm{N}(1125, 369)$
Find $R > 1140$	$\mathrm{P}(R > 1200) = \mathrm{P}(Z > \frac{1140-1125}{\sqrt{369}})$ $= \mathrm{P}(Z > 0.7809)$ $= 1 - 0.7823$ $= 0.2177$

Example 3: The two independent random variables, X and Y have the distributions $X \sim \mathrm{N}(16, 3)$ and $Y \sim \mathrm{N}(41, 4)$. Find $\mathrm{P}(3X > Y)$.

Find the distribution of $3X - Y$.	$aX + bY \sim \mathrm{N}(a\mu_1 + b\mu_2, a^2\sigma_1{}^2 + b^2\sigma_2{}^2)$ $\mu = 3(16) - 41$ $= 7$ $\sigma^2 = 3^2(3) + 4$ $= 31$ $3X - Y \sim \mathrm{N}(7, 31)$
Find $\mathrm{P}(3X - Y > 0)$.	$\mathrm{P}(3X - Y > 0) = \mathrm{P}(Z > \frac{0-7}{\sqrt{31}})$ $= \mathrm{P}(Z > -1.257)$ $= 0.8962$

Combining Random Variables with Identical Normal Distribution

When combining multiple variables which have the same normal distribution, the mean and standard deviation can be found by simply multiplying the mean and variation of one variable by the number of variables combined.

For random independent variables $X_1, X_2, X_3, X_4 \ldots X_n$ with identical distribution $X \sim \mathrm{N}(\mu_1, \sigma_1{}^2)$,

$$\sum_{i=1}^{n} X_i \sim \mathrm{N}(n\mu_1, n\sigma_1{}^2)$$

Example 4: i) The random variables $X_1, X_2, X_3, X_4, X_5, X_6$ and X_7 all have a distribution of $X \sim \mathrm{N}(58, 4)$. Find the distribution of the random variable R, when

$$R = \sum_{i=1}^{7} X_i$$

Multiply the mean and standard variation by n where $n = 7$.	Mean=7(58) $= 406$ Variance $= 7(4)$ $= 28$
	The distribution of R is: $R \sim \mathrm{N}(406, 28)$